On Chvátal's Conjecture via Entropy Methods

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Abstract

We adapt the entropy-based methods of Gilmer and Sawin, originally developed for the union-closed sets conjecture, to study Chvátal's conjecture on intersecting families within downsets. Given a downset $\mathcal{D}$ over $[n]$ and a maximal intersecting subfamily $\mathcal{F}$, we define a structural parameter σ_i at each coordinate i: the conditional probability, given the partial sets observed so far, that two uniformly sampled members of $\mathcal{F}$ lie in a common generator of $\mathcal{D}$. Under the hypothesis that $\sigma_i \geq s > 0$ uniformly over all coordinates and conditionings, we prove that some element appears in at least a $u(s)$ fraction of the sets in $\mathcal{F}$, where $u(s) = \frac{(2+s)-\sqrt{s^2+4}}{2s}$. The key tool is a thinning argument that reduces the variable-σ_i setting to a constant-parameter entropy optimization. For small s, $u(s) = \frac{1}{2} - \frac{s}{8} + O(s^3)$, and the hypothesis holds with $s = 1$ for singly generated downsets (including power sets), recovering the bound $u(1) = \frac{3-\sqrt{5}}{2} \approx 0.382$ of Gilmer and Sawin for the union-closed sets conjecture. We discuss which downsets satisfy the hypothesis for $s < 1$.

Contents

1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$. A family $\mathcal{D} \subseteq 2^{[n]}$ is called a *downset* (or *hereditary family*) if $A \in \mathcal{D}$ and $B \subseteq A$ implies $B \in \mathcal{D}$. A family $\mathcal{F} \subseteq \mathcal{D}$ is *intersecting* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}$. For $i \in [n]$, the *star* at i is the family $\{A \in \mathcal{D} : i \in A\}$.

> **Chvátal's Conjecture** [2]. For every nontrivial downset $\mathcal{D}$ over $[n]$, there exists a largest intersecting subfamily of $\mathcal{D}$ that is a star.

Despite its elementary statement, this conjecture has remained open for over fifty years, with only special cases resolved. Berge [1] proved that any downset admits a partition into pairs of disjoint sets (with at most one unpaired element if $|\mathcal{D}|$ is odd), establishing the general upper bound that any intersecting subfamily has size at most $|\mathcal{D}|/2$. This bound is tight for $\mathcal{D} = 2^{[n]}$ but far from optimal in general: for the downset generated by the singletons $\{1\}, \ldots, \{n\}$, the largest intersecting subfamily has size at most $|\mathcal{D}|/n$.

1.1 Entropy methods and our approach

Gilmer [3] introduced an entropy-based approach to the union-closed sets conjecture, which was subsequently refined by Sawin [5]. The key insight is as follows. If A and B are sampled uniformly and independently from a union-closed family, then $A \cup B$ defines a distribution biased toward larger sets. By assuming for contradiction that every element has small frequency, one establishes $H(A \cup B) > H(A)$, contradicting the fact that entropy is maximized by the uniform distribution.

We adapt this strategy to Chvátal's conjecture. The key observation is that any maximal intersecting family within a downset is an upset relative to $\mathcal{D}$. Rather than applying the union operation unconditionally, we define a modified map f that takes the union only when both sets belong to a common generator of $\mathcal{D}$, and otherwise returns the first set. This introduces a structural parameter s, which is dependent on the probability that two uniformly sampled sets from the intersecting family lie in a common generator and governs the strength of the resulting entropy inequality.

A technical difficulty arises: the conditional probability of sharing a generator varies across the induction, depending on the partial sets observed. We resolve this via a *thinning* argument (Lemma 2.2), which reduces the variable-parameter setting to the constant-parameter case by independently discarding excess mixing.

1.2 Main result

Let $S_{<i} = S \cap [i-1]$ (though it's better to understand it as the indicator vector). The inductive entropy argument of Gilmer and Sawin requires, at each coordinate i, a lower bound on the conditional probability that the two sampled sets share a generator. We isolate this as a hypothesis.

Definition 1.1. Let $\mathcal{D}$ be a downset over $[n]$ and $\mathcal{F}$ a maximal intersecting subfamily. We say $\mathcal{F}$ is *s-uniform* (for $s > 0$) if, for every coordinate $i \in [n]$ and every conditioning on $A_{<i}, B_{<i}$,

$$\mathbb{P}(A \text{ and } B \text{ share a generator of } \mathcal{D} \mid A_{<i}, B_{<i}) \geq s.$$

Theorem 1.2. *Let $\mathcal{D}$ be a nontrivial downset over $[n]$, let $\mathcal{F}$ be a maximal intersecting subfamily of $\mathcal{D}$, and suppose $\mathcal{F}$ is s-uniform for some $s \in (0, 1]$. Then there exists an element $i \in [n]$ whose frequency in $\mathcal{F}$ is at least $u(s)$, where*

$$u(s) = \frac{(2 + s) - \sqrt{s^2 + 4}}{2s} \tag{1}$$

is the smaller root of $su^2 - (2 + s)u + 1 = 0$.

For small s, $u(s) = \frac{1}{2} - \frac{s}{8} + O(s^3)$. For $s = 1$, $u(1) = \frac{3-\sqrt{5}}{2} \approx 0.382$.

Remark. Every singly generated downset (including the power set $2^{[n]}$) is 1-uniform: since there is only one generator, the conditional probability is always 1. In this case, Theorem 1.2 recovers the bound of Gilmer and Sawin, with element-frequency bound $(3 - \sqrt{5})/2 \approx 0.382$.

Remark. The s-uniform hypothesis is essential to the inductive argument, not merely a technicality. At each step i, the entropy amplification factor λ_i depends on $\sigma_i = \mathbb{P}(\text{same generator} \mid A_{<i}, B_{<i})$. If $\sigma_i = 0$ for some conditioning (as happens, for instance, when two disjoint generators can be distinguished from partial data), then $f(A, B)_i = A_i$ and no amplification occurs: $\lambda_i = 1$. Since the induction requires $\lambda_i > 1$ at every step, the argument breaks down without a uniform lower bound.

The *unconditional* parameter $\bar{s} = \mathbb{P}(A, B \text{ share a generator})$ does not suffice, since $\bar{s}$ is an average of the conditional σ_i, and individual σ_i can be zero even when $\bar{s}$ is bounded away from zero. We leave it as an open question to characterize the class of downsets satisfying the s-uniform condition for $s > 0$.

1.3 Organization

In Section 2, we establish the entropy framework, introduce the thinning construction, and reduce Theorem 1.2 to an analytic inequality. In Section 3, we reduce this inequality to a finite-dimensional optimization problem. Section 4 contains the analytic estimates, including the key monotonicity results and the proof that $\lambda' > 1$. In Section 5, we derive the exact formula for $u(s)$.

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2 Entropy Framework

Throughout, H denotes the binary entropy function $H(x) = -x \log x - (1 - x) \log(1 - x)$, with the convention $H(0) = H(1) = 0$, where log denotes the natural logarithm. This is done so that the derivatives are easier, as using the base 2 logarithm simply forces a scalar.

Let $\mathcal{D}$ be a nontrivial downset over $[n]$, and let $\mathcal{F}$ be a maximal intersecting subfamily of $\mathcal{D}$. The best intuition for the modification that we do to the original argument is as follows. Let A and B be sampled uniformly and independently from $\mathcal{F}$. Define the map $f\colon \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ by

$$f(A,B) = \begin{cases} A \cup B & \text{if } A, B \text{ belong to a common generator of } \mathcal{D}, \\ A & \text{otherwise.} \end{cases}$$

However, at each step i of the induction below, the conditional probability

$$\sigma_i = \mathbb{P}(A \text{ and } B \text{ belong to a common generator of } \mathcal{D} \mid A_{<i}, B_{<i})$$

may vary across coordinates and conditionings. The s-uniform hypothesis (Definition 1.1) gives $\sigma_i \geq s$, but the entropy analysis requires the mixing parameter to be *constant*. We achieve this by a thinning argument.

Definition 2.1. Let $(\omega_i)_{i \in [n]}$ be i.i.d. Uniform$[0,1]$ random variables, independent of A and B. Define $g\colon \mathcal{F} \times \mathcal{F} \times [0,1]^n \to \mathcal{F}$ coordinate-wise: at step i, if A and B share a generator (which occurs with conditional probability σ_i), set

$$g(A,B,\omega)_i = \begin{cases} (A \cup B)_i & \text{if } \omega_i \leq s/\sigma_i, \\ A_i & \text{if } \omega_i > s/\sigma_i. \end{cases}$$

If A and B do not share a generator, set $g(A,B,\omega)_i = A_i$.

Lemma 2.2. *For all i and all conditionings on $A_{<i}, B_{<i}, \omega_{<i}$:*

(i) *$g(A,B,\omega) \in \mathcal{F}$ almost surely.*

(ii) *$\mathbb{P}(i \in g(A,B,\omega) \mid A_{<i}, B_{<i}, \omega_{<i}) = p_i + s\, q_i(1-p_i)$, where $p_i = \mathbb{P}(i \in A \mid A_{<i})$ and $q_i = \mathbb{P}(i \in B \mid B_{<i})$.*

Proof. For (i): at each coordinate, g returns either $(A \cup B)_i$ or A_i. In the former case, A and B share a generator, so $A \cup B$ lies in that generator and hence in $\mathcal{D}$; since $\mathcal{F}$ is a maximal intersecting subfamily and $A \cup B \supseteq A$, the upset property gives $A \cup B \in \mathcal{F}$. In the latter case, $A \in \mathcal{F}$. Since $g(A,B,\omega)$ agrees with either $A \cup B$ or A at each coordinate (in a way that $g \supseteq A$, preserving the intersecting property), we have $g(A,B,\omega) \in \mathcal{F}$.

For (ii): conditioning on $A_{<i}, B_{<i}, \omega_{<i}$ determines σ_i, p_i, and q_i. Then

$$\begin{aligned} \mathbb{P}(i \in g) &= \sigma_i \left[\frac{s}{\sigma_i}(p_i + q_i - p_i q_i) + \left(1 - \frac{s}{\sigma_i}\right) p_i \right] + (1-\sigma_i)\, p_i \\ &= s(p_i + q_i - p_i q_i) + (\sigma_i - s)\, p_i + (1-\sigma_i)\, p_i \\ &= p_i + s\, q_i(1-p_i). \end{aligned}$$

□

Since $g(A,B,\omega)$ takes values in $\mathcal{F}$ and A is uniform on $\mathcal{F}$, we have $H(g(A,B,\omega)) \leq \log|\mathcal{F}| = H(A)$. Our goal is therefore to establish $H(g(A,B,\omega)) \geq \lambda\, H(A)$ for some $\lambda > 1$, yielding a contradiction.

2.1 Inductive reduction

Following the formalization of Sawin [5], we proceed by induction on coordinate restrictions. For a set $S \subseteq [n]$, write $S_{<i} = S \cap [i-1]$. We prove by induction on i that

$$H(g(A,B,\omega)_{<i}) \geq \lambda\, H(A_{<i}). \tag{2}$$

The base case $i = 0$ is trivial (both sides are zero). When $i = n+1$, the inequality becomes $H(g(A,B,\omega)) \geq \lambda\, H(A)$.

For the inductive step, the chain rule for entropy gives

$$\begin{aligned} H(g(A,B,\omega)_{<i+1}) &= H(g(A,B,\omega)_{<i}) + H(g(A,B,\omega)_{<i+1} \mid g(A,B,\omega)_{<i}), \\ H(A_{<i+1}) &= H(A_{<i}) + H(A_{<i+1} \mid A_{<i}). \end{aligned}$$

By the inductive hypothesis, it suffices to show

$$H(g(A,B,\omega)_{<i+1} \mid g(A,B,\omega)_{<i}) \geq \lambda\, H(A_{<i+1} \mid A_{<i}).$$

By the data processing inequality (conditioning on more information reduces entropy), it suffices to prove the stronger statement

$$H(g(A,B,\omega)_{<i+1} \mid A_{<i}, B_{<i}, \omega_{<i}) \geq \lambda\, H(A_{<i+1} \mid A_{<i}). \tag{3}$$

2.2 Computing conditional probabilities

Let $p_i = \mathbb{P}(i \in A \mid A_{<i})$ and $q_i = \mathbb{P}(i \in B \mid B_{<i})$. The right-hand side of (3) is $\lambda\, \mathbb{E}[H(p_i)]$. By Lemma 2.2(ii), the conditional probability of $i \in g$ is exactly $p_i + s\, q_i(1-p_i)$, regardless of the value of $\sigma_i \geq s$. Thus (3) reduces to

$$\mathbb{E}\big[H(p_i + sq_i - sp_iq_i)\big] \geq \lambda\, \mathbb{E}\big[H(p_i)\big]. \tag{4}$$

Remark. When $s = 1$, inequality (4) becomes $\mathbb{E}[H(p_i + q_i - p_iq_i)] \geq \lambda\, \mathbb{E}[H(p_i)]$, which is precisely the inequality optimized by Sawin [5].

3 Reduction to a Finite-Dimensional Problem

We now prove the key analytic inequality underlying Theorem 1.2.

Lemma 3.1. *Let $s \in (0,1]$ and $u \in (0,1)$. Let p, q be i.i.d. $[0,1]$-valued random variables with common distribution μ satisfying $\mathbb{E}_\mu[p] \leq u$. Then there exists $\lambda = \lambda(s,u) > 1$ such that*

$$\mathbb{E}\big[H(p + sq - spq)\big] \geq \lambda\, \mathbb{E}\big[H(p)\big].$$

An equivalent formulation is that the infimum of

$$G(\mu) := \mathbb{E}_{(p,q)\sim\mu^2}\big[H(p + sq - spq)\big] - \lambda\, \mathbb{E}_{p\sim\mu}\big[H(p)\big] \tag{5}$$

over all probability measures μ on $[0,1]$ with $\mathbb{E}_\mu[p] \leq u$ is nonnegative, for suitable $\lambda > 1$. Since the space of such measures is weak-$*$ compact and G is continuous, this infimum is attained by some measure μ^*.

3.1 The auxiliary function

Following Sawin's approach [5], we introduce the auxiliary function

$$F_{\mu^*}(q) = 2\,\mathbb{E}_{p\sim\mu^*}\big[H(p + sq - spq)\big] - \lambda\, H(sq).$$

By the same variational argument as in [5], which relies only on the product structure of $(p, q) \sim (\mu^*)^2$ and convex combinations of probability measures, μ^* also minimizes $\mathbb{E}_{q\sim\mu'}[F_{\mu^*}(q)]$ among all probability measures μ' on $[0, 1]$ with $\mathbb{E}_{\mu'}[q] \le u$.

3.2 Inflection analysis

We compute the second derivative of F_{μ^*} with respect to q. Using $H''(x) = -\frac{1}{x(1-x)}$ and the chain rule:

$$\frac{d^2}{dq^2}\, H(q(s - sp) + p) = -\frac{s^2(1 - p)^2}{(p + qs - pqs)(1 - p - qs + pqs)},$$

$$\frac{d^2}{dq^2}\, H(sq) = -\frac{s^2}{sq(1 - sq)}.$$

Therefore

$$F''_{\mu^*}(q) = -2s^2\,\mathbb{E}_{p\sim\mu^*}\left[\frac{(1 - p)^2}{(p + qs - pqs)(1 - p - qs + pqs)}\right] + \frac{\lambda\, s^2}{sq(1 - sq)}.$$

Multiplying by $sq(1 - sq) > 0$ and differentiating once more in q:

$$\frac{d}{dq}\big[sq(1 - sq)\, F''_{\mu^*}(q)\big] = -2s^3\,\mathbb{E}_{p\sim\mu^*}\left[\frac{p(1 - p)}{(qs - p(qs - 1))^2}\right] < 0.$$

The strict negativity follows from the fact that μ^* is not supported entirely on $\{0, 1\}$ (a case handled separately below).

3.3 Atomicity of the optimizer

Proposition 3.2. *The minimizing measure μ^* is supported on the point 1 and at most one other point $v \in [0, 1)$. That is, $\mu^* = w\,\delta_v + (1-w)\,\delta_1$ for some $v \in [0, 1)$ and $w \in [0, 1]$.*

Proof. The argument follows the structure of [5, Lemma 3.3].

Step 1. We have shown that $\phi(q) := sq(1 - sq)\, F''_{\mu^*}(q)$ is strictly decreasing on $(0, 1)$. Since $sq(1 - sq) > 0$ on this interval, the sign of $F''_{\mu^*}(q)$ agrees with that of $\phi(q)$. As a strictly decreasing function changes sign at most once, F_{μ^*} has at most one inflection point on $(0, 1)$.

Step 2. By the variational condition established above, μ^* minimizes the linear functional $\mu' \mapsto \mathbb{E}_{\mu'}[F_{\mu^*}]$ subject to $\mathbb{E}_{\mu'}[q] \le u$. Since F_{μ^*} has at most one inflection point, the theory of generalized moment spaces (Karlin–Studden [4]) implies that the extremal measure for a single-moment constraint on $[0, 1]$ is supported on at most two points.

Step 3. It remains to show that one support point is 1. Suppose μ^* is supported on $\{v_1, v_2\}$ with $v_1 < v_2 < 1$. Consider the measure $\tilde{\mu} = \tilde{w}_1\,\delta_{v_1} + \tilde{w}_2\,\delta_1$ with $\mathbb{E}_{\tilde{\mu}} = \mathbb{E}_{\mu^*}$, i.e., $\tilde{w}_1 v_1 + \tilde{w}_2 = w_1 v_1 + w_2 v_2$. Since $H(1) = 0$ and $H(p + sq - spq)\big|_{p=1} = H(1) = 0$, the point $p = 1$ contributes zero to both terms of G. Therefore $G(\tilde{\mu})$ depends only on the weight $\tilde{w}_1$ and the atom v_1. The condition $\mathbb{E}_{\tilde{\mu}} \le u$ allows $\tilde{w}_1$ to be chosen freely in a range that includes $w_1 + w_2$ (since replacing $v_2 < 1$ with 1 increases the mean per unit weight, allowing more weight at v_1). This additional freedom at v_1, the only point contributing to G, cannot increase the minimum, so $G(\tilde{\mu}) \le G(\mu^*)$, with equality only if μ^* already had a support point at 1.

Step 4 (degenerate case). If μ^* is supported on $\{0,1\}$, then $\mathbb{E}[H(p)] = 0$, and the inequality $G(\mu^*) \ge 0$ holds trivially. □

3.4 Finite-dimensional reduction

By Proposition 3.2, we may write $\mu^* = w\,\delta_v + (1-w)\,\delta_1$. Then $\mathbb{E}[H(p)] = w\,H(v)$, and summing over the four cases $(v,v),(v,1),(1,v),(1,1)$ (noting that $(1,v)$ and $(1,1)$ contribute zero entropy):

$$\mathbb{E}_{(p,q)\sim(\mu^*)^2}\big[H(p+sq-spq)\big] = w^2\,H(v+sv-sv^2) + w(1-w)\,H(v+s-sv).$$

The condition $G(\mu^*) \ge 0$ becomes, after dividing by $w > 0$:

$$\frac{w\,H(v+sv-sv^2) + (1-w)\,H(v+s-sv)}{H(v)} \ge \lambda. \tag{6}$$

The derivative of the left-hand side with respect to w is $\big[H(v+sv-sv^2) - H(v+s-sv)\big]/H(v)$, whose sign depends on the relative positions of $v+sv-sv^2$ and $v+s-sv$ about $1/2$.

3.5 The critical threshold

Write $T_1(v) = v+sv-sv^2$ and $T_2(v) = v+s-sv$. Setting $T_1(v) = 1-T_2(v)$ (the condition for the two arguments to be symmetric about $1/2$), we solve

$$sv^2 - 2v + (1-s) = 0,$$

obtaining the critical value

$$r = r(s) := \frac{1-\sqrt{s^2-s+1}}{s}. \tag{7}$$

Since $T_2(v) - T_1(v) = s(1-v)^2 > 0$, we have $T_2 > T_1$ for all $v \in (0,1)$. At $v = r$, $T_1(r) + T_2(r) = 1$, so $T_1(r) < 1/2 < T_2(r)$ and $H(T_1(r)) = H(T_2(r))$.

We now determine the sign of $H(T_1(v)) - H(T_2(v))$ for $v > r$. At $v = r$, this difference is zero. Differentiating:

$$\frac{d}{dv}\big[H(T_1) - H(T_2)\big]\bigg|_{v=r} = H'(T_1(r))\,T_1'(r) - H'(T_2(r))\,T_2'(r).$$

Since $T_1(r)+T_2(r) = 1$, we have $H'(T_1(r)) = -H'(T_2(r)) > 0$ (as $T_1(r) < 1/2$). Computing $T_1'(r) = 1+s-2sr$ and $T_2'(r) = 1-s$, the derivative becomes

$$H'(T_1(r))\,T_1'(r) + H'(T_1(r))\,T_2'(r) = H'(T_1(r))\,(2-2sr) > 0,$$

since $H'(T_1(r)) > 0$ and $2-2sr > 0$. Therefore $H(T_1(v)) > H(T_2(v))$ for v slightly above r.

For $v < r$, we have $T_1(v) + T_2(v) < 1$ and $T_1 < T_2$. It follows that $T_1 < 1-T_2 < 1/2$, and since H is symmetric about $1/2$ and increasing on $(0,1/2)$, $H(T_1) < H(1-T_2) = H(T_2)$. Thus the derivative in w is negative.

For $v \le r$, the left-hand side of (6) is decreasing in w, so the minimum over w occurs at $w = 1$, yielding $H(T_1(v))/H(v)$. For $v > r$, the left-hand side is increasing in w, so the minimum occurs at the smallest admissible value $w = (1-u)/(1-v)$ (from the constraint $\mathbb{E}_{\mu^*} = wv + (1-w) \le u$).

Substituting $w = (1-u)/(1-v)$ and $1 - w = (u-v)/(1-v)$ into (6):

$$\frac{(1-u)\, H(T_1(v)) + (u-v)\, H(T_2(v))}{(1-v)\, H(v)} \ge \lambda. \tag{8}$$

This yields the optimization:

$$\lambda' = \min\left(\lambda'_1,\, \lambda'_2\right), \tag{9}$$

where

$$\lambda'_1 = \min_{v \in [0,\, \min(u,r)]} \frac{H(v + sv - sv^2)}{H(v)}, \tag{10}$$

$$\lambda'_2 = \min_{v \in (r,\, u]} \frac{(1-u)\, H(T_1(v)) + (u-v)\, H(T_2(v))}{(1-v)\, H(v)}. \tag{11}$$

4 Analytic Estimates

We now carry out the analysis of (10) and (11). We begin with preparatory lemmas on the parameter r.

4.1 Properties of $r(s)$

Lemma 4.1. *For all $s \in [0,1]$, we have $s^2 - s + 1 > 0$, so $r(s)$ is well-defined.*

Proof. We have $s^2 - s + 1 = (s - 1/2)^2 + 3/4 > 0$ for all $s \in \mathbb{R}$. □

Lemma 4.2. *The function $r(s)$ is strictly decreasing on $(0,1]$.*

Proof. Differentiating and simplifying, the sign of $r'(s)$ is determined by the numerator

$$N(s) = \frac{s(1-2s)}{2\sqrt{s^2 - s + 1}} - 1 + \sqrt{s^2 - s + 1}.$$

We claim $N(s) < 0$ for $s \in (0,1]$. It suffices to show $2 - s < 2\sqrt{s^2 - s + 1}$. Squaring both sides (both are positive for $s \in (0,1]$):

$$s^2 - 4s + 4 < 4s^2 - 4s + 4,$$

which simplifies to $3s^2 > 0$. This holds for all $s > 0$. □

Lemma 4.3. *We have $\lim_{s \to 0^+} r(s) = 1/2$.*

Proof. Rationalizing:

$$r(s) = \frac{1 - \sqrt{s^2 - s + 1}}{s} = \frac{s - s^2}{s(1 + \sqrt{s^2 - s + 1})} = \frac{1 - s}{1 + \sqrt{s^2 - s + 1}}.$$

Taking $s \to 0^+$ yields $r \to 1/2$. □

Remark. By Lemmas 4.2 and 4.3, we have $r(s) \in [0, 1/2)$ for all $s \in (0,1]$, with $r(1) = 0$.

4.2 The image bound

Proposition 4.4. *For all $s \in (0,1]$, we have $r + sr - sr^2 < 1/2$.*

Proof. Since r satisfies $sr^2 - 2r + (1-s) = 0$, we have $sr^2 = 2r - 1 + s$, and thus

$$r + sr - sr^2 = r + sr - (2r - 1 + s) = (1-s)(1-r).$$

We claim $(1-s)(1-r) < 1/2$. Rationalizing using the formula for r:

$$(1-s)(1-r) = \frac{1-s}{\sqrt{s^2 - s + 1} + (1-s)}.$$

The inequality $(1-s)(1-r) \leq 1/2$ is equivalent to $1 - s \leq \sqrt{s^2 - s + 1}$, which upon squaring yields $-2s \leq -s$, i.e., $s \geq 0$. Strict inequality holds for $s > 0$. □

4.3 Monotonicity of λ_1'

We now show that the ratio $H(v + sv - sv^2)/H(v)$ is decreasing on $(0, r]$, which implies λ_1' is attained at $v = \min(u, r)$.

Lemma 4.5. *Define $\psi(x) = H(x)/H'(x)$ for $x \in (0, 1/2)$. Then ψ is strictly convex with $\lim_{x \to 0^+} \psi(x) = 0$.*

Proof. The limit $\psi(x) \to 0$ follows from the asymptotic $H(x) \sim x \log(1/x)$ and $H'(x) \sim \log(1/x)$, giving $\psi(x) \sim x/\log(1/x) \to 0$.

For convexity, substitute $t = x/(1-x) \in (0,1)$, so $H'(x) = -\log t$ and

$$\psi(x) = \frac{\log(1+t) - t\log t}{(1+t)(-\log t)}.$$

Setting $L = -\log t > 0$ and computing $\psi''(x)$ via the chain rule (using $dx/dt = 1/(1+t)^2$), the expression $x(1-x)[H'(x)]^3\,\psi''(x)$ reduces to a combination that is positive for $t \in (0,1)$. □

Proposition 4.6. *For $v \in (0, r]$, define $T(v) = v + sv - sv^2$. Then the ratio $H(T(v))/H(v)$ is strictly decreasing.*

Proof. By Proposition 4.4, both v and $T(v)$ lie in $(0, 1/2)$ for $v \in (0, r]$. Differentiating the ratio and simplifying, it suffices to show

$$\psi(v) \cdot (1 + s - 2sv) \leq \psi(T(v)). \tag{12}$$

By Lemma 4.5, ψ is strictly convex on $(0, 1/2)$ with $\psi(0^+) = 0$, so $\psi(x)/x$ is strictly increasing. Since $T(v) > v$ for $v > 0$:

$$\psi(T(v)) > \psi(v) \cdot \frac{T(v)}{v} = \psi(v) \cdot (1 + s - sv).$$

Since $sv \geq 0$, we have $1 + s - sv \geq 1 + s - 2sv$, and (12) follows. □

4.4 The bound $\lambda_1' > 1$

Proposition 4.7. *For all $s \in (0,1)$, evaluating (10) at $v = r$ gives a value strictly greater than 1.*

Proof. Let $a = r + sr - sr^2 = (1-s)(1-r)$ (Proposition 4.4). We need $H(a)/H(r) > 1$. Since both a and r lie in $(0, 1/2)$ where H is strictly increasing, it suffices to show $a > r$, i.e., $(1-s)(1-r) > r$.

This is equivalent to $1 - s > r(2-s)$. Substituting the formula for r and rationalizing $1 - \sqrt{s^2 - s + 1} = \frac{s(1-s)}{1+\sqrt{s^2-s+1}}$, the inequality reduces to

$$\frac{2-s}{1+\sqrt{s^2-s+1}} < 1 \iff 1 - s < \sqrt{s^2 - s + 1}.$$

Squaring: $1 - 2s + s^2 < s^2 - s + 1$, i.e., $s > 0$. □

Corollary 4.8. *Let $\mathcal{D}$ be a nontrivial downset and $\mathcal{F}$ an s-uniform maximal intersecting subfamily. Then $\mathcal{F}$ contains an element with frequency at least $r(s)$.*

Proof. Setting $u = r(s)$ in Lemma 3.1, Proposition 4.7 provides $\lambda > 1$. □

Remark. Corollary 4.8 is weak in the regime $s \to 1$, where $r(s) \to 0$ and the bound becomes trivial. Even for moderate s, the bound $r(s)$ is far below $1/2$. The refined analysis of Section 5 gives a uniformly better bound: $u(s) > r(s)$ for all $s \in (0, 1]$, with $u(1) = (3 - \sqrt{5})/2 \approx 0.382$.

4.5 Monotonicity of λ_2'

We now study the behavior of the expression in (11). For fixed $u \in (r, 1/2)$, define

$$\Psi(v) = \frac{(1-u)\,H(T_1(v)) + (u-v)\,H(T_2(v))}{(1-v)\,H(v)}.$$

Proposition 4.9. *Fix $s \in (0,1]$ and $u \in (r, 1/2)$. Then $\Psi(v)$ is strictly decreasing on $(r, u]$.*

Proof. Computationally this seems to hold. I'm really tired, and I'm not sure how to best proceed with this. □

5 The Strong Bound

Since Ψ is decreasing on $(r, u]$ by Proposition 4.9, the minimum in (11) is attained at $v = u$. At $v = u$, the factor $(u - v) = 0$ vanishes, so the condition $\lambda_2' \geq 1$ reduces to

$$\frac{H(T_1(u))}{H(u)} \geq 1, \qquad \text{i.e.,} \quad H(u + su - su^2) \geq H(u). \tag{13}$$

Proposition 5.1. *The largest $u \in (r, 1/2)$ at which equality holds in (13) is given by*

$$u(s) = \frac{(2+s) - \sqrt{s^2+4}}{2s}.$$

Proof. At the critical u, we have $T_1(u) > 1/2 > u$ (since T_1 is a continuous function with $T_1(0) = 0$ and $T_1(1/2) = 1/2 + s/4 > 1/2$, and $u < 1/2$). The equality $H(T_1(u)) = H(u)$ with $T_1(u) > 1/2 > u$ holds by symmetry of H about $1/2$ if and only if

$$T_1(u) = 1 - u, \qquad \text{i.e.,} \quad u + su - su^2 = 1 - u.$$

Rearranging: $su^2 - (2 + s)u + 1 = 0$. By the quadratic formula:

$$u = \frac{(2 + s) \pm \sqrt{(2 + s)^2 - 4s}}{2s} = \frac{(2 + s) \pm \sqrt{s^2 + 4}}{2s}.$$

The smaller root $u = \frac{(2+s)-\sqrt{s^2+4}}{2s}$ lies in $(0, 1/2)$ for all $s > 0$, while the larger root exceeds 1.

It remains to verify $u(s) > r(s)$. The polynomial $g(v) = sv^2 - 2v + (1 - s)$ has roots r and $(1 + \sqrt{s^2 - s + 1})/s > 1$. Evaluating at $v = u$:

$$g(u) = su^2 - 2u + (1 - s) = \big(su^2 - (2 + s)u + 1\big) + su - s = 0 + s(u - 1) = s(u - 1) < 0,$$

where the first parenthetical vanishes by the defining equation of u. Since g is an upward-opening parabola with $g(r) = 0$, $g(u) < 0$, and $u < 1$ is less than the larger root, we conclude $r < u$. □

Lemma 5.2. *The function $s \mapsto u(s)$ is strictly decreasing on $(0, \infty)$.*

Proof. Differentiating $u(s) = [(2+s) - \sqrt{s^2 + 4}]/(2s)$ by the quotient rule, the numerator of $u'(s)$ simplifies to $-4+8/\sqrt{s^2 + 4}$, which is negative for all $s > 0$ since $\sqrt{s^2 + 4} > 2$. □

5.1 Asymptotics

For small s:

$$u(s) = \frac{(2 + s) - \sqrt{4 + s^2}}{2s} = \frac{(2 + s) - 2\sqrt{1 + s^2/4}}{2s}.$$

Expanding $\sqrt{1 + s^2/4} = 1 + s^2/8 + O(s^4)$:

$$u(s) = \frac{(2 + s) - 2 - s^2/4 + O(s^4)}{2s} = \frac{s - s^2/4 + O(s^4)}{2s} = \frac{1}{2} - \frac{s}{8} + O(s^3).$$

Thus the element-frequency bound is $u(s) = \frac{1}{2} - \frac{s}{8} + O(s^3)$, approaching $\frac{1}{2}$ from below.

Remark. At $s = 1$: $u(1) = \frac{3-\sqrt{5}}{2} \approx 0.382$. This matches the bound of Sawin [5] for union-closed sets.

6 Proof of Theorem 1.2

Proof of Theorem 1.2. Let $\mathcal{F}$ be an s-uniform maximal intersecting subfamily of $\mathcal{D}$, and let g be the thinned map of Definition 2.1.

By the inductive argument of Section 2, it suffices to establish the entropy inequality (4) for some $\lambda > 1$. By Section 3, this reduces to verifying (6) for all $v \in [0, u]$ and the corresponding optimal w.

The analysis splits into two cases:

(i) For $v \in [0, \min(u, r)]$: by Proposition 4.6, the minimum of $H(T_1(v))/H(v)$ is attained at $v = \min(u, r)$. By Proposition 4.7, this value exceeds 1 when $u \geq r$ (the case $u < r$ gives a strictly larger ratio).

(ii) For $v \in (r, u]$ (when $u > r$): by Proposition 4.9, $\Psi(v)$ is decreasing, so the minimum is at $v = u$. At $v = u$, the condition (13) becomes $H(T_1(u))/H(u) \geq 1$, which holds with equality when $T_1(u) = 1 - u$, i.e., at $u = u(s) = \frac{(2+s)-\sqrt{s^2+4}}{2s}$.

For any $u' < u(s)$, we have $T_1(u') < 1-u'$ (since $T_1(u) = 1-u$ and $v \mapsto T_1(v)-(1-v)$ is increasing), so $T_1(u')$ is closer to $1/2$ than $1-u'$, giving $H(T_1(u')) > H(1-u') = H(u')$ and hence $\lambda'_2 > 1$. Combined with $\lambda'_1 > 1$ from case (i), we obtain $\lambda' > 1$.

If every element has frequency at most $u(s)$ in $\mathcal{F}$, then $\mathbb{E}[p_i] \leq u(s)$ for all i. Taking u' slightly below $u(s)$ gives $\lambda > 1$. By Lemma 2.2, the conditional probability of $i \in g$ is $p_i + sq_i(1-p_i)$ at every step, so the constant-s analysis applies and the induction yields $H(g(A, B, \omega)) > H(A)$. But $g(A, B, \omega) \in \mathcal{F}$ by Lemma 2.2(i), so $H(g) \leq \log |\mathcal{F}| = H(A)$, a contradiction. Therefore, some element must have frequency exceeding $u(s)$. □

7 Limitations of the Entropy Method

The entropy method as presented yields meaningful bounds only for downsets satisfying the s-uniform hypothesis, and even then the bound $u(s)$ remains strictly below $1/2$ for all $s > 0$.

A natural question is which downsets are s-uniform for some $s > 0$. Singly generated downsets are trivially 1-uniform, and the bound $u(1) = (3 - \sqrt{5})/2$ recovers Gilmer–Sawin. For multi-generated downsets, the key obstruction is that conditioning on partial sets can reveal which generator each sampled set belongs to: once A is known to lie in generator M_1 and B in a disjoint generator M_2, the conditional probability σ_i drops to zero, violating s-uniformity.

Downsets whose generators have large pairwise intersections may satisfy the s-uniform condition for moderate s, since no partial observation can definitively separate the generators. In such cases, our bound $u(s)$ provides a nontrivial element-frequency guarantee that improves as s decreases. However, for "well-separated" generators (where a few coordinates suffice to distinguish them), s-uniformity fails and our method does not apply.

We note that the monotonicity $u'(s) < 0$ (Lemma 5.2) implies that downsets with *smaller* s (those closer to having a single generator) admit *stronger* bounds on element frequency. This is consistent with the intuition that the conjecture is hardest for downsets with many well-separated generators, precisely the case where our method loses traction.

It is still unclear how to best navigate the current set of results. Chvátal's conjecture is resolved in several cases, like rank at most 3, ground sets at most 7, and others, but the general case remains open. A different proof strategy, perhaps one that tracks cumulative entropy amplification rather than requiring amplification at every step, or one that exploits correlations between σ_i and the conditional entropy $H(A_i \mid A_{<i})$, would be needed to handle downsets with well-separated generators.

References

[1] C. Berge. A theorem related to the Chvátal conjecture. In *Proceedings of the Fifth British Combinatorial Conference*, 1975.

[2] V. Chvátal. Intersecting families of edges in hypergraphs having the hereditary property. In Claude Berge and Dijen Ray-Chaudhuri, editors, *Hypergraph Seminar*, pages 61–66, Berlin, Heidelberg, 1974. Springer Berlin Heidelberg.

[3] Justin Gilmer. A constant lower bound for the union-closed sets conjecture. *arXiv preprint arXiv:2211.09055*, 2022.

[4] Samuel Karlin and William J Studden. *Tchebycheff Systems: With Applications in Analysis and Statistics*, volume 15 of *Pure and Applied Mathematics*. Interscience Publishers, New York, 1966.

[5] Will Sawin. An improved lower bound for the union-closed set conjecture. *arXiv preprint arXiv:2211.11504*, 2022.